## The Mass-Radius Relation for Polytropes

The mass within any point of a polytropic star is

$$\mathcal{M} = \int_0^R 4\pi r^2 \rho \, dr = 4\pi r_n^3 \rho_c \int_0^{\xi'} \xi^2 \theta^n d\xi \qquad (16.1.1)$$

This can be simplified by substituting the left side of the Lane-Emden equation for $\theta^n$

$$\mathcal{M}(\xi') = 4\pi r_n^3 \rho_c \int_0^{\xi'} -\frac{d}{d\xi}\left(\xi^2 \frac{d\theta}{d\xi}\right) d\xi = 4\pi r_n^3 \rho_c \, {\xi'}^2 \left(\frac{d\theta}{d\xi}\right)_{\xi'}$$

$$= 4\pi \left[\frac{(n+1)K}{4\pi G}\right]^{\frac{3}{2}} \rho_c^{\frac{3-3n}{2n}+1} \, {\xi'}^2 \left(-\frac{d\theta}{d\xi}\right)_{\xi'}$$

$$= 4\pi \left[\frac{(n+1)K}{4\pi G}\right]^{\frac{3}{2}} \rho_c^{\frac{(3-n)}{2n}} \, {\xi'}^2 \left(-\frac{d\theta}{d\xi}\right)_{\xi'} \qquad (16.1.2)$$

Now compare this to the radius

$$r(\xi') = r_n \xi' = \left[\frac{(n+1)K}{4\pi G}\right]^{\frac{1}{2}} \rho_c^{\frac{(1-n)}{2n}} \xi' \qquad (16.1.3)$$

If you solve for $\rho_c$ in the mass equation (16.1.2)

$$\rho_c = \left\{ \left[\frac{\mathcal{M}}{4\pi}\right] \left[\frac{4\pi G}{(n+1)K}\right]^{\frac{3}{2}} \left[-{\xi'}^2 \left(\frac{d\theta}{d\xi}\right)_{\xi'}\right]^{-1} \right\}^{\frac{2n}{3-n}}$$

and eliminate it from the radius equation (16.1.3), you get

$$r = (4\pi)^{\frac{1}{n-3}} \left[\frac{(n+1)K}{G}\right]^{\frac{n}{3-n}} \left[-{\xi'}^2 \frac{d\theta}{d\xi}\right]^{\frac{n-1}{3-n}} \mathcal{M}^{\frac{1-n}{3-n}} \qquad (16.1.4)$$

In other words, the radius of a polytropic star is related to its mass by

$$R \propto \mathcal{M}_T^{(1-n)/(3-n)}$$

Note what this means: for a polytropic index of 3/2 (the $\gamma = 5/3$ case), $R \propto \mathcal{M}^{-1/3}$. Thus, for a set of stars with the same $K$ and $n$ (*i.e.*, white dwarfs, or a fully convective star that is undergoing mass transfer), the stellar radius is inversely proportional to the mass.

Equation (16.1.4) can also be re-written to give the polytropic constant, $K$, in terms of the star's total mass and radius. The expression is

$$K = \left[ \frac{4\pi}{\xi^{n+1}} \left( -\frac{d\theta}{d\xi} \right)^{1-n} \right]_{\xi_1}^{1/n} \frac{G}{n+1} \mathcal{M}_T^{1-1/n} R^{-1+3/n} \qquad (16.1.5)$$

## The Chandrasekhar Mass Limit

For $n = 3$, $(\gamma = 4/3)$, the total mass of a polytropic star is independent of its central density. This is important, for if you substitute in the equation of state for a completely relativistic electron gas ($K = 1.2435 \times 10^{15}\,\mu^{-4/3}$ cgs) into (16.1.2), and do the math, the total mass of the star becomes

$$\mathcal{M}_{\rm Ch} = \frac{5.82}{\mu_e^2}\mathcal{M}_\odot \qquad (16.1.6)$$

This is the Chandrasekhar mass limit. For $\mathcal{M} < \mathcal{M}_{\rm Ch}$, a star can adjust its structure and back away from the completely relativistic limiting case. However, since (16.1.4) already represents the limit to electron pressure, degenerate stars cannot exceed $\mathcal{M}_{\rm Ch}$. If we assume such a star has no hydrogen, then, by (5.1.6), $\mu_e = 2$, and $\mathcal{M}_{\rm Ch} = 1.456\mathcal{M}_\odot$.

## The Central Temperature and Density

With a little algebra, equations (16.1.2) and (16.1.3) can be manipulated to yield the conditions in the center of a polytropic star. By combining the two equations, while eliminating $K$, the expression for central density as a function of $\mathcal{M}_T$ and $R$ becomes

$$\rho_c = \frac{1}{4\pi}\,\xi_1 \left(-\frac{d\theta}{d\xi}\right)^{-1}_{\xi_1} \frac{\mathcal{M}_T}{R^3} \tag{16.2.1}$$

The central pressure then follows, by combining this with the equation for $K$ given in (16.1.5)

$$P_c = K\rho_c^{1+1/n} = \frac{1}{4\pi(n+1)} \left(-\frac{d\theta}{d\xi}\right)^{-2}_{\xi_1} \frac{G\mathcal{M}_T^2}{R^4} \tag{16.2.2}$$

Finally, the central temperature can be found from (16.2.1), (16.2.2), and the ideal gas law

$$T_c = \frac{\mu\, m_a}{k}\frac{P_c}{\rho_c} = \frac{\mu\, m_a}{(n+1)k} \left(-\xi\frac{d\theta}{d\xi}\right)^{-1}_{\xi_1} \frac{G\mathcal{M}_T}{R} \tag{16.2.3}$$

Numerically, these expressions evaluate to

$$\rho_c = 0.47\,\xi_1 \left(-\frac{d\theta}{d\xi}\right)^{-1}_{\xi_1} \left(\frac{\mathcal{M}_T}{\mathcal{M}_\odot}\right)\left(\frac{R}{R_\odot}\right)^{-3} \quad \text{g cm}^{-3}$$

$$P_c = \frac{8.97\times 10^{14}}{(n+1)} \left(-\frac{d\theta}{d\xi}\right)^{-2}_{\xi_1} \left(\frac{\mathcal{M}_T}{\mathcal{M}_\odot}\right)^2\left(\frac{R}{R_\odot}\right)^{-4} \quad \text{dyne cm}^{-2}$$

$$T_c = \frac{2.29\times 10^{7}}{(n+1)} \left(-\xi\,\frac{d\theta}{d\xi}\right)^{-1}_{\xi_1} \left(\frac{\mathcal{M}_T}{\mathcal{M}_\odot}\right)\left(\frac{R}{R_\odot}\right)^{-1} \quad \text{K}$$

## The Hayashi Track

We can locate polytropic-like stars in the HR diagram by matching fully convective models to boundary conditions for cool, convective stars. To do this, we first assume that the star is entirely convective, except for a thin radiative region at the surface, and that the star is cool, so that $H^-$ opacity dominates. From our boundary condition analysis

$$P_t = 2^{2/3} P_p \tag{9.26}$$

Now recall that the pressure at the photosphere is

$$P_p = \frac{2}{3} \frac{G\mathcal{M}_T}{R^2 \kappa} \tag{9.12}$$

and that $\kappa$ itself is a function of pressure

$$\kappa = \kappa_0' P^\alpha T^\beta$$

So,

$$P_p = \frac{2}{3} \frac{G\mathcal{M}_T}{R^2} \frac{1}{\kappa_0' P_p^\alpha T_{\rm eff}^\beta} \quad \Longrightarrow \quad P_p^{\alpha+1} = \frac{2}{3} \frac{G\mathcal{M}_T}{R^2 \, \kappa_0' \, T_{\rm eff}^\beta}$$

or, if we substitute for $R$ using $\mathcal{L}_T = 4\pi R^2 \sigma T_{\rm eff}^4$,

$$P_p = \left\{ \frac{2}{3} \frac{4\pi\sigma}{\kappa_0'} \frac{G\mathcal{M}_T}{\mathcal{L}_T} \, T_{\rm eff}^{4-\beta} \right\}^{1/(\alpha+1)} \tag{16.3.1}$$

The pressure at the top of the convective polytrope is therefore

$$P_t = c_1 \left( \frac{\mathcal{M}_T}{\mathcal{L}_T} \right)^{1/(\alpha+1)} T_{\rm eff}^{(4-\beta)/(\alpha+1)} \tag{16.3.2}$$

where

$$c_1 = 2^{2/3} \left( \frac{2}{3} \frac{4\pi\sigma G}{\kappa_0'} \right)^{1/(\alpha+1)} \tag{16.3.3}$$

Now, consider the polytropic equation

$$P = K\rho^{1+1/n} = \left( \frac{k}{\mu\, m_a} \right)^{n+1} K^{-n}\, T^{n+1} \tag{16.3.4}$$

where the constant $K$ is given by (16.1.5), *i.e.*,

$$\begin{aligned} K^{-n} &= \frac{1}{4\pi} \left( -\xi^{n+1} \frac{d\theta}{d\xi} \right)_{\xi_1} \left( \frac{n+1}{G} \right)^n \mathcal{M}_T^{1-n} R^{n-3} \\ &= \frac{1}{4\pi} \left( -\xi^{n+1} \frac{d\theta}{d\xi} \right)_{\xi_1} \left( \frac{n+1}{G} \right)^n \left( \frac{\mathcal{L}_T}{4\pi\sigma} \right)^{\frac{(n-3)}{2}} \mathcal{M}_T^{1-n}\, T_{\text{eff}}^{6-2n} \end{aligned}$$

At the top of the convective layer,

$$T_t = \left( \frac{8}{5} \right)^{2/9} T_{\text{eff}} \tag{9.24}$$

so if we substitute this in (16.3.4), the pressure becomes

$$P_t = c_2\, \mathcal{M}_T^{1-n} \mathcal{L}_T^{(n-3)/2}\, T_{\text{eff}}^{7-n} \tag{16.3.5}$$

where

$$c_2 = \left( \frac{k}{\mu\, m_a} \right)^{n+1} \left( -\xi^{n+1} \frac{d\theta}{d\xi} \right)_{\xi_1} \left( \frac{n+1}{G} \right)^n \left( \frac{8}{5} \right)^{\frac{2n+2}{9}} \frac{(4\pi\sigma)^{\frac{3-n}{2}}}{4\pi}$$

If we now set (16.3.2) equal to (16.3.5), the relation between $\mathcal{M}_T$, $\mathcal{L}_T$, and $T_{\text{eff}}$ for fully convective stars becomes

$$T_{\text{eff}}^{7-n+\frac{\beta-4}{\alpha+1}} = \left(\frac{c1}{c2}\right) \mathcal{M}_T^{n-1+\frac{1}{\alpha+1}} \, \mathcal{L}_T^{\frac{3-n}{2}-\frac{1}{\alpha+1}} \tag{16.3.6}$$

Or, for the case of fully convective ($n = 3/2$) stars with $H^-$ opacity ($\alpha = 1/2$, $\beta = 17/2$),

$$T_{\text{eff}} \sim 2600 \, \mu^{13/51} \left(\frac{\mathcal{M}_T}{\mathcal{M}_\odot}\right)^{7/51} \left(\frac{\mathcal{L}_T}{\mathcal{L}_\odot}\right)^{1/102} \text{K}$$

This is the Hayashi track (although, due to our boundary condition assumptions, the constant is a factor of 1.5 too small). Note that the temperature is nearly independent of luminosity, and is only weakly dependent on mass. All fully convective stars will therefore fall in the same cool area of the HR diagram.